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On the quantum BRST structure of classical mechanics.

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Abstract

The BRST-antiBRST invariant path integral formulation of classical mechanics of Gozzi et al is generalized to pseudomechanics. It is shown that projections to physical propagators may be obtained by BRST-antiBRST invariant boundary conditions. The formulation is also viewed from recent group theoretical results within BRST-antiBRST invariant theories. A natural bracket expressed in terms of BRST and antiBRST charges in the extended formulation is shown to be equal to the Poisson bracket. Several remarks on the operator formulation are made.

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1 Introduction.

Gozzi *et al* have in a series of papers developed a path integral formulation of classical mechanics [1]-[4]. More precisely they have given a BRST-antiBRST quantum formulation of classical mechanics in which $\hbar$ is strictly one. In this paper their treatment is further generalized and interpreted from the BFV-formulation and viewed from recent general group theoretic results for BRST-antiBRST theories. In section 2 the path integrals for classical mechanics is generalized to pseudoclassical mechanics in which some coordinates are odd Grassmann numbers. In section 3 it is pointed out that there is also an operator BRST formulation of classical mechanics, and it is shown how the ghost extended propagators are reduced to propagators in classical mechanics by BRST-antiBRST invariant boundary conditions. In section 4 the Hamiltonian flow is viewed from recent general group theoretical results for BRST-antiBRST theories. These results contain a new Poisson bracket, the Q-bracket, defined in terms of the BRST and antiBRST charges which is shown to yield the correct Poisson bracket for classical mechanics. The general group theoretical results also contain extended group transformations which here are given explicitly for the Hamiltonian flow. In section 5 some remarks on quantization are given. In two appendices a further extended BRST-antiBRST algebra is given and the properties of the Q-brackets are displayed.

2 Path integrals for pseudoclassical mechanics.

In [3] the path integral formulation of classical mechanics was generalized to classical mechanics in arbitrary coordinates on a symplectic manifold. Here we further generalize this treatment to pseudoclassical mechanics in arbitrary coordinates on a supersymplectic manifold. We consider real coordinates ϕ^a with Grassmann parities $\varepsilon_a \equiv \varepsilon(\phi^a) = 0, 1$. In terms of super or graded Poisson bracket (PB) we have

$$\{\phi^a, \phi^b\} = \omega^{ab}(\phi). \tag{2.1}$$

Hamilton's equations are

$$\dot{\phi}^a = \{\phi^a, H(\phi)\} = \omega^{ab}\partial_b H(\phi), \tag{2.2}$$

where $H(\phi)$ is a real Hamiltonian. The dot denotes derivative with respect to time t. All equations are local in t. The coordinates ϕ^a are assumed to span the manifold which means that $\omega^{ab}(\phi)$ has an inverse $\omega_{ab}(\phi)$ satisfying the properties

$$\omega_{ab}\omega^{bc} = \omega^{cb}\omega_{ba} = \delta^c_a, \quad \varepsilon(\omega_{ab}) = \varepsilon(\omega^{ab}) = \varepsilon_a + \varepsilon_b. \tag{2.3}$$

ω_{ab} are coefficients of the basic two-form Ω:

$$\Omega = d\phi^b \wedge d\phi^a \, \omega_{ab} = \omega_{ab} d\phi^b \wedge d\phi^a (-1)^{\varepsilon_a + \varepsilon_b}. \tag{2.4}$$

Since Ω is closed ($d\Omega = 0$) ω_{ab} must satisfy

$$\partial_a \omega_{bc}(-1)^{(\varepsilon_a+1)\varepsilon_c} + cycle(a, b, c) = 0, \tag{2.5}$$

which is equivalent to

$$\omega^{ad}\partial_d\omega^{bc}(-1)^{\varepsilon_a\varepsilon_c} + cycle(a, b, c) = 0, \tag{2.6}$$

which are the jacobi identities of (2.1). Note the symmetry properties

$$\omega^{ab}=-\omega^{ba}(-1)^{\varepsilon_a\varepsilon_b}, \quad \omega_{ab}=\omega_{ba}(-1)^{(\varepsilon_a+1)(\varepsilon_b+1)}. \tag{2.7}$$

The derivation of the corresponding path integral formulation follows now from the treatment in [1]-[4]. We may define the propagator by [3]

$$P(\phi,t;\phi_0,0)=\delta(\phi-\phi_{cl}(\phi_0,t)), \tag{2.8}$$

where ϕ_{cl} is the solution of (2.2). The crucial steps in the derivation of the path integral formulation are first to view the delta function as a functional delta function and then to rewrite it as follows:

$$\delta(\phi-\phi_{cl})=\delta(\dot{\phi}^a-\omega^{ab}\partial_b H(\phi))\mathrm{sdet}(\delta^a_b\partial_t-\partial_b(\omega^{ac}\partial_c H(\phi)), \tag{2.9}$$

where 'sdet' is the superdeterminant or the Berezian. Rewriting the delta function in terms of integration over Lagrange multipliers λ_a and expressing the superdeterminant as integrations over ghost variables leads then to the path integral formulation

$$P(\phi,t;\phi_0,0)=\int D\lambda D\mathcal{P}D\mathcal{C}\exp\{i\int_0^t L_{eff}\}, \tag{2.10}$$

where the effective Lagrangian is

$$L_{eff}=\lambda_a\dot{\phi}^a+\mathcal{P}_a\dot{\mathcal{C}}^a-\lambda_a\omega^{ab}\partial_b H-\mathcal{P}_a\mathcal{C}^b\partial_b(\omega^{ac}\partial_c H)(-1)^{\varepsilon_a+\varepsilon_b}. \tag{2.11}$$

Compared to a path integral in quantum mechanics, $\hbar$ is strictly one in (2.10). In (2.10) we have introduced the Lagrange multipliers λ_a, $\varepsilon(\lambda_a)=\varepsilon_a$, and the ghosts $\mathcal{C}^a$ and $\mathcal{P}_a$ with Grassmann parities, $\varepsilon(\mathcal{C}^a)=\varepsilon(\mathcal{P}_a)=\varepsilon_a+1$. ($\mathcal{P}_a$ corresponds to $i\bar{\mathcal{C}}_a$ in the notation of [1]-[4].) The reality properties are

$$(\phi^a)^*=\phi^a, \quad (\lambda_a)^*=\lambda_a(-1)^{\varepsilon_a}, \quad (\mathcal{C}^a)^*=-\mathcal{C}^a(-1)^{\varepsilon_a}, \quad (\mathcal{P}_a)^*=\mathcal{P}_a, \tag{2.12}$$

and $H^*=H$. These reality properties imply that L_{eff} is real. (Notice that they imply $(\omega^{ab})^*=-\omega^{ba}=\omega^{ab}(-1)^{\varepsilon_a\varepsilon_b}$ and $(\partial_a B)^*=B^*\overleftarrow{\partial}_a$.) Since the Lagrangian (2.11) is given in a standard phase space form we may immediately read off the effective Hamiltonian,

$$H_{eff}=\lambda_a\omega^{ab}\partial_b H+\mathcal{P}_a\mathcal{C}^b\partial_b(\omega^{ac}\partial_c H)(-1)^{\varepsilon_a+\varepsilon_b}, \tag{2.13}$$

and that (ϕ^a,λ_a) and $(\mathcal{C}^a,\mathcal{P}_a)$ are canonically conjugate pairs. The fundamental nonzero elements of the extended PB are therefore

$$\{\phi^a,\lambda_b\}=\delta^a_b, \quad \{\mathcal{C}^a,\mathcal{P}_b\}=\delta^a_b, \tag{2.14}$$

which are consistent with the reality properties (2.12). (Notice that this bracket is different from (2.1) since $\{\phi^a,\phi^b\}=0$ here.) The resulting equations of motion are

$$\begin{aligned}
\dot{\phi}^a&=\{\phi^a,H_{eff}\}=\omega^{ab}\partial_b H,\\
\dot{\lambda}_a&=\{\lambda_a,H_{eff}\}=-\lambda_b\partial_a(\omega^{bc}\partial_c H)(-1)^{\varepsilon_a}-\mathcal{P}_c\mathcal{C}^b\partial_a\partial_b(\omega^{cd}\partial_d H)(-1)^{\varepsilon_a+\varepsilon_b+\varepsilon_c},\\
\dot{\mathcal{C}}^a&=\{\mathcal{C}^a,H_{eff}\}=\mathcal{C}^b\partial_b(\omega^{ac}\partial_c H)(-1)^{\varepsilon_a+\varepsilon_b},\\
\dot{\mathcal{P}}_a&=\{\mathcal{P}_a,H_{eff}\}=-\mathcal{P}_b\partial_a(\omega^{dc}\partial_c H)(-1)^{(\varepsilon_b+1)\varepsilon_a},
\end{aligned} \tag{2.15}$$

which also follows from the Lagrangian (2.11). Notice that the original equation for ϕ^a is retained. In fact this equation cannot be derived from a Lagrangian in the original coordinates ϕ^a unless the two-form Ω (2.4) is exact (see appendix A).

The effective action $S_{eff} = \int dt L_{eff}(t)$ is invariant under the BRST transformations (ε is here an odd constant)

$$\begin{aligned} &\delta\phi^a = \varepsilon\mathcal{C}^a(-1)^{\varepsilon_a}, \quad \delta\lambda_a = \delta\mathcal{C}^a = 0,\\ &\delta\mathcal{P}_a = -\varepsilon\lambda_a, \end{aligned} \tag{2.16}$$

as well as under the antiBRST transformations

$$\begin{aligned} &\bar{\delta}\phi^a = \varepsilon\mathcal{P}_b\omega^{ba}, \quad \bar{\delta}\lambda_a = -\varepsilon\mathcal{P}_b\partial_a\omega^{bc}\lambda_c(-1)^{\varepsilon_a+\varepsilon_c+\varepsilon_a\varepsilon_b},\\ &\bar{\delta}\mathcal{C}^a = \varepsilon\lambda_b\omega^{ba}(-1)^{\varepsilon_a} + \varepsilon\mathcal{P}_b\mathcal{C}^c\partial_c\omega^{ba}(-1)^{\varepsilon_a+\varepsilon_b+\varepsilon_c},\\ &\bar{\delta}\mathcal{P}_a = \frac{1}{2}\varepsilon\mathcal{P}_c\mathcal{P}_b\partial_a\omega^{bc}(-1)^{\varepsilon_a(\varepsilon_b+\varepsilon_c+1)+\varepsilon_c}. \end{aligned} \tag{2.17}$$

These transformations are generated by the BRST and antiBRST charges

$$Q = \mathcal{C}^a\lambda_a, \quad \bar{Q} = -\mathcal{P}_a\lambda_b\omega^{ba} - \frac{1}{2}\mathcal{P}_a\mathcal{P}_b\mathcal{C}^c\partial_c\omega^{ba}(-1)^{\varepsilon_b+\varepsilon_c} \tag{2.18}$$

in terms of the extended PB (2.14). One may easily show that

$$\{Q, Q\} = \{Q, \bar{Q}\} = \{\bar{Q}, \bar{Q}\} = 0, \tag{2.19}$$

where the last equality requires the Jacobi identities (2.6) of the original PB (2.1). Obviously there is a relation between the original PB (2.1) and the extended PB (2.14) with the BRST and the antiBRST charges. This connection will be given in section 4 and appendix B. That the BRST and antiBRST charges, Q and $\bar{Q}$, are conserved follows from the fact that the effective Hamiltonian (2.13) may be written as follows

$$H_{eff} = -\{Q, \{\bar{Q}, H\}\} = \{\bar{Q}, \{Q, H\}\}. \tag{2.20}$$

The symmetry algebra (2.19) may be further extended to the following algebra (cf.[1]-[4])[2]

$$\begin{aligned} &\{Q, Q\} = \{Q, \bar{Q}\} = \{\bar{Q}, \bar{Q}\} = 0,\\ &\{\bar{K}, Q\} = \bar{Q}, \quad \{K, \bar{Q}\} = Q,\\ &\{\bar{K}, K\} = Q_g, \quad \{\bar{K}, \bar{Q}\} = \{K, Q\} = 0,\\ &\{Q, Q_g\} = Q, \quad \{\bar{Q}, Q_g\} = -\bar{Q},\\ &\{K, Q_g\} = 2K, \quad \{\bar{K}, Q_g\} = -2\bar{K}, \end{aligned} \tag{2.21}$$

where

$$Q_g \equiv \mathcal{P}_a\mathcal{C}^a \tag{2.22}$$

is the ghost charge, and

$$K \equiv \mathcal{C}^a\omega_{ab}\mathcal{C}^b(-1)^{\varepsilon_a}, \quad \bar{K} \equiv \mathcal{P}_a\omega^{ab}\mathcal{P}_b. \tag{2.23}$$

[2]Such an algebra was first given for Yang-Mills theories in [5], and for general Lie group theories within the BFV-scheme in [6].

Notice the different symmetry properties of ω_{ab} and ω^{ab} (2.7). Only K involves the inverse of ω^{ab}, *i.e.* K exists only if ω^{ab} is nondegenerate. Furthermore, the relation $\{K, \bar{Q}\} = Q$ requires the Jacobi identities in the form (2.5). One may easily check that

$$\{Q_g, H_{eff}\} = \{K, H_{eff}\} = \{\bar{K}, H_{eff}\} = 0. \tag{2.24}$$

The reality properties from (2.12) are that $\bar{Q}$ and Q_g are real while Q, K, and $\bar{K}$ are imaginary. (H_{eff} is real.)

3 Interpretations

The BRST-antiBRST formulation of classical mechanics given in the previous section is a quantum gauge theory in which $\hbar = 1$. The BRST charge Q has the standard BFV-form [7] which tells us that we have an abelian gauge theory where λ_a are the original gauge generators. They make the phase space coordinates ϕ^a completely arbitrary. In fact, all variables are unphysical which means that the theory has no physical quantum degrees of freedom at all, which of course is the reason why it may describe classical mechanics. The BFV-form of the effective Lagrangian (2.11) is

$$L_{eff} = \lambda_a \dot{\phi}^a + \mathcal{P}_a \dot{\mathcal{C}}^a - \{Q, \psi\}, \tag{3.1}$$

where the gauge fixing fermion ψ is

$$\psi = \{H, \bar{Q}\} = \mathcal{P}_a \omega^{ab} \partial_b H. \tag{3.2}$$

BFV prescribes the form $\psi = \mathcal{P}_a \chi^a$ where χ^a are gauge fixing variables to λ_a. Here we have $\chi^a = \omega^{ab} \partial_b H$ which is an unconventional choice. Consistency requires the matrix

$$\{\lambda_a, \chi^b\} = -\partial_a(\omega^{bc} \partial_c H)(-1)^{\varepsilon_a} \tag{3.3}$$

to be invertible, a condition which may be compared to the regularity of the superdeterminant in the basic derivation (2.9). This condition requires the original Hamiltonian H to be regular. If H is not regular then even the original theory is a gauge theory or at least a constrained theory and the effective theory given here is no longer appropriate. Notice that H is only introduced through the choice of gauge fixing, which means that the time evolution of classical mechanics is determined by the choice of gauge fixing. Indeed from a Lagrangian point of view the entire Hamilton's equation (2.2) is introduced through a choice of gauge fixing [4].

It is possible to transform the above path integral formulation of classical mechanics into an operator formulation. One has then to impose the canonical commutation relations from (2.14)

$$[\phi^a, \lambda_b] = i\delta^a_b, \quad [\mathcal{C}^a, \mathcal{P}_b] = i\delta^a_b, \tag{3.4}$$

where the commutators are graded commutators defined by

$$[A, B] = AB - BA(-1)^{\varepsilon_A \varepsilon_B}. \tag{3.5}$$

$\hbar$ is of course strictly one. The corresponding operator charges, Q, $\bar{Q}$, K, $\bar{K}$ are still given by (2.18) and (2.23). However, the ghost charge operator Q_g is given by (cf.(2.22))

$$Q_g = \frac{1}{2}(\mathcal{P}_a \mathcal{C}^a - \mathcal{C}^a \mathcal{P}_a (-1)^{\varepsilon_a}). \tag{3.6}$$

$\bar{Q}$ and Q_g are hermitian while Q, K, and $\bar{K}$ are antihermitian. Their algebra is given by (2.21) with PB's replaced by commutators multiplied by $(-i)$. The effective Hamiltonian operator is

$$H_{eff}=[Q,[\bar{Q},H]]=-\partial_b H\omega^{ba}\lambda_a(-1)^{\varepsilon_a+\varepsilon_b}-\mathcal{C}^b\mathcal{P}_a\partial_b(\omega^{ac}\partial_c H)(-1)^{\varepsilon_a\varepsilon_b}, \quad (3.7)$$

which at the 'classical' level is equal to (2.13).

It is natural to define propagators by

$$\langle\phi',\mathcal{C}'|e^{-itH_{eff}}|\phi,\mathcal{C}\rangle=\int D\lambda D\mathcal{P}D\mathcal{C}\exp\{i\int_0^t L_{eff}\}, \quad (3.8)$$

where $|\phi,\mathcal{C}\rangle$ are eigenstates to ϕ^a and $\mathcal{C}^a$, and where L_{eff} is the Lagrangian (2.11). Such propagators were also considered in [1]-[4] and were used to consider topological aspects in [4]. Classical mechanics should emerge from BRST-antiBRST invariant boundary conditions. However, one may notice that the boundary conditions $\mathcal{C}^a=\mathcal{C}'^a=0$ are BRST invariant, and $\mathcal{P}_a=\mathcal{P}'_a=0$ antiBRST invariant. Thus, in order to have both BRST and antiBRST invariant boundary conditions both these conditions should be imposed which is impossible since $\mathcal{C}^a$ and $\mathcal{P}_a$ are canonically conjugate variables. On the other hand it seems possible to impose the asymmetric boundary conditions $\mathcal{C}^a=\mathcal{P}'_a=0$ or $\mathcal{C}'^a=\mathcal{P}_a=0$ in which case we obtain the physical propagators ($\mathcal{P}_a=0$ is equal to an integration over $\mathcal{C}^a$)

$$P(\phi',t;\phi,0)=\int d\mathcal{C}\langle\phi',\mathcal{C}'=0|e^{-itH_{eff}}|\phi,\mathcal{C}\rangle, \quad (3.9)$$

or

$$P(\phi',t;\phi,0)=\int d\mathcal{C}'\langle\phi',\mathcal{C}'|e^{-itH_{eff}}|\phi,\mathcal{C}=0\rangle, \quad (3.10)$$

both of which correspond to the original derivation in (2.9) and (2.10) read backwards. These projections are peculiar since we impose BRST invariant boundary conditions on one side and antiBRST invariant conditions on the other. Now it is also possible to impose strictly BRST-antiBRST invariant boundary conditions provided the set $\{\phi^a\}$ contains an even number of both odd and even coordinates which actually is what the standard BFV-prescription requires. (In a recent paper [8] projections of extended propagators to physical ones were studied in details mainly within the standard BFV-formulation. The prescriptions given there are also applicable to the present case.) In this case we have to perform a polarization, *i.e.* a split in coordinates and momenta. We should therefore first make a transformation to Darboux coordinates. However, to simplify matters we assume that ω^{ab} is constant in the following. We may then split the phase space variables into coordinates and momenta as follows

$$\phi^a=(x^\alpha,p^\alpha), \quad \omega^{ab}=\begin{pmatrix} 0, & \eta^{\alpha\beta} \\ -\eta^{\alpha\beta}, & 0 \end{pmatrix}, \quad (3.11)$$

where $a,b=1,\ldots,2n$ and $\alpha,\beta=1,\ldots,n$ and where $\eta^{\alpha\beta}$ is a constant symmetric metric ($\eta^{\alpha\beta}=\eta^{\beta\alpha}(-1)^{\varepsilon_\alpha\varepsilon_\beta}$). We split also the remaining variables

$$\lambda_a=(\lambda_\alpha,\rho_\alpha), \quad \mathcal{C}^a=(\mathcal{C}^\alpha,\bar{\mathcal{C}}^\alpha), \quad \mathcal{P}_a=(\mathcal{P}_\alpha,\bar{\mathcal{P}}_\alpha). \quad (3.12)$$

The BRST and antiBRST charges (2.18) may then be written as follows

$$Q = \mathcal{C}^\alpha\lambda_\alpha + \bar{\mathcal{C}}^\alpha\rho_\alpha, \quad \bar{Q} = -\bar{\mathcal{P}}_\alpha\lambda_\beta\eta^{\beta\alpha} + \mathcal{P}_\alpha\rho_\beta\eta^{\beta\alpha}. \tag{3.13}$$

We are now naturally led to the following two simple BRST-antiBRST invariant states

$$\begin{aligned} &\rho_\alpha|\psi\rangle_1 = \bar{\mathcal{P}}_\alpha|\psi\rangle_1 = \mathcal{C}^\alpha|\psi\rangle_1 = 0,\\ &\lambda_\alpha|\psi\rangle_2 = \mathcal{P}_\alpha|\psi\rangle_2 = \bar{\mathcal{C}}^\alpha|\psi\rangle_2 = 0. \end{aligned} \tag{3.14}$$

If we define wave functions by $\psi(x,p,\mathcal{C},\bar{\mathcal{C}}) \equiv \langle x,p,\mathcal{C},\bar{\mathcal{C}}|\psi\rangle$ where $|x,p,\mathcal{C},\bar{\mathcal{C}}\rangle$ are the same eigenstates as in (3.8), then the wave function representation of the states $|\psi\rangle_{1,2}$ in (3.14) are $\psi_1 = \delta(\mathcal{C})\phi_1(x)$ and $\psi_2 = \delta(\bar{\mathcal{C}})\phi_2(p)$, where $\phi_{1,2}$ may be gauge fixed to $\phi_1(x) = \delta(x - x_{cl})$ and $\phi_2(p) = \delta(p - p_{cl})$. Notice that $\phi_1(x)$ and $\phi_2(p)$ are not Fourier transforms of each other since x^α and p^α are not canonically conjugate variables here. The point object may therefore be localized in both coordinate and momentum space, a property which only is valid in classical mechanics. From the above results we may now directly derive the appropriate BRST-antiBRST invariant boundary conditions for the projections of the extended propagator (3.8) to physical ones: We have the coordinate space propagator relevant for ψ_1 given by

$$P(x',t;x,0) = \int dp'dpd\bar{\mathcal{C}}'d\bar{\mathcal{C}}\langle x',p',\bar{\mathcal{C}}',\mathcal{C}' = 0|e^{-itH_{eff}}|x,p,\bar{\mathcal{C}},\mathcal{C} = 0\rangle, \tag{3.15}$$

and the momentum space propagator relevant for ψ_2 given by

$$P(p',t;p,0) = \int dx'dxd\mathcal{C}'d\mathcal{C}\langle x',p',\bar{\mathcal{C}}' = 0,\mathcal{C}'|e^{-itH_{eff}}|x,p,\bar{\mathcal{C}} = 0,\mathcal{C}\rangle. \tag{3.16}$$

The above expressions are a little bit heuristic: Firstly, we should have used eigenstates to the ghost momenta, $\mathcal{P}_a$, since they are hermitian operators (see (2.12)). Then we should have used indefinite metric so that half of the hermitian operators have imaginary eigenvalues, and the operator $e^{-itH_{eff}}$ should be replaced by the hermitian operator $e^{-tH_{eff}}$ [8].

4 Group theory within the BRST-antiBRST framework

In [9] general group theory within a BRST-antiBRST framework was studied. This was done at the quantum operator level. The formulas may however easily be rewritten in terms of classical functions. A group transformed function $A(\psi)$ where ψ^α are the group parameters were first assumed to satisfy the Lie equations

$$A(\psi)\overleftarrow{\nabla}_\alpha \equiv A(\psi)\overleftarrow{\partial}_\alpha - \{A(\psi), Y_\alpha(\psi)\} = 0, \tag{4.1}$$

where $\partial_\alpha \equiv \partial/\partial\psi^\alpha$, and where the connections Y_α were assumed to be of the form

$$Y_\alpha(\psi) = \{Q, \{\bar{Q}, X_\alpha(\psi)\}\}. \tag{4.2}$$

Q and $\bar{Q}$ are the BRST-antiBRST charges respectively. The integrability conditions for X_α following from (4.1) are

$$X_\alpha\overleftarrow{\partial}_\beta - X_\beta\overleftarrow{\partial}_\alpha(-1)^{\varepsilon_\alpha\varepsilon_\beta} - \{X_\alpha, X_\beta\}_Q = \{X_{\alpha\beta}, Q\} + \{Y_{\alpha\beta}, \bar{Q}\}, \tag{4.3}$$

where the new Q-bracket is defined by

$$\{A, B\}_Q \equiv \frac{1}{2}\Big(\{\{A, \bar{Q}\}, \{Q, B\}\} - \{\{A, Q\}, \{\bar{Q}, B\}\}\Big). \tag{4.4}$$

This bracket satisfies all properties of a graded PB except for the Jacobi identities and Leibniz' rule (see appendix B). It is nondegenerate on one-fourth of the unphysical part of the extended supersymplectic manifold. In the present case all variables are unphysical and one easily finds that

$$\{\phi^a, \phi^b\}_Q = \omega^{ab}(\phi). \tag{4.5}$$

Thus, on the original supersymplectic manifold the Q-bracket (4.4) is equal to the original PB (2.2). Only the original coordinates ϕ^a span the Q-bracket (see appendix B). (In [2] another multiPB expression involving Q, K and $\bar{K}$ was shown to yield the original bracket for ω^{ab} constant.) Further properties of the Q-bracket for the present case are given in appendix B.

Consider now the Lie equations (4.1) and their integrability conditions (4.3). For Lie group theories $X_{\alpha\beta}$ and $Y_{\alpha\beta}$ in (4.3) may be chosen to be zero. This is also the case in the present case since we have an abelian gauge theory. The group aspect is in fact only the Hamiltonian flow. The only group parameter is time t. The Lie equations (4.1) reduces therefore to Hamilton's equation

$$\dot{A}(t) - \{A(t), H_{eff}\} = 0. \tag{4.6}$$

For the choices $X_\alpha \equiv \phi^a$, $X_\beta \equiv H$ in (4.3), the integrability conditions reduce to the original Hamilton's equation

$$\dot{\phi}^a = \{\phi^a, H\}_Q = \omega^{ab}\partial_b H. \tag{4.7}$$

This shows the generality of the connections between (4.6) and (4.7).

In [9] the integrability conditions (4.3) for X_α as well as $X_{\alpha\beta}$, $Y_{\alpha\beta}$ etc were shown to be possible to embed into a master equation. A general solution of this master equation was then given. This solution is unfortunately not directly applicable to the present case. Firstly, the construction assumes that the group generators are connected to λ_a in the BRST charge and not just the Hamiltonian flow. Secondly, the explicit construction is made within a manifest Sp(2) representation of the BRST and antiBRST charges using a different ghost number representation in which both Q and $\bar{Q}$ have ghost number plus one [10]. Only in the case when ω^{ab} is constant are Q and $\bar{Q}$ in a manifest Sp(2) form here. However, even in this case the appropriate ghost numbers are impossible to determine without an explicit form of H (H has the new ghost number minus two).

Although we cannot demonstrate the details of the Sp(2)-construction of [9] in the present case, we may extract the final consequences of that paper. In [9] it was shown that one naturally arrives at an extended formulation in which the number of group parameters are quadrupled to a supersymmetric set. In this extended framework there are then extended forms of the BRST-antiBRST charges. In the present case the time parameter t is replaced by $t,\ \rho,\ \theta,\ \bar{\theta}$ where $t,\ \rho$ are even and $\theta,\ \bar{\theta}$ odd. Furthermore, one has to introduce conjugate momenta to these variables and define a further extended PB. We have the conjugate pairs (t, π), (ρ, σ), (θ, ξ) and $(\bar{\theta}, \bar{\xi})$ satisfying

$$\{t, \pi\}_e = \{\rho, \sigma\}_e = \{\theta, \xi\}_e = \{\bar{\theta}, \bar{\xi}\}_e = 1. \tag{4.8}$$

On this extended super symplectic manifold we may introduce the extended BRST and antiBRST charges Δ and $\bar{\Delta}$ given by

$$\Delta \equiv Q + \theta\pi + \rho\bar{\xi}, \quad \bar{\Delta} \equiv \bar{Q} + \bar{\theta}\pi - \rho\xi. \tag{4.9}$$

We have also the following extended charges

$$R \equiv K + \theta\bar{\xi}, \quad \bar{R} \equiv \bar{K} + \bar{\theta}\xi, \quad \tilde{Q}_g \equiv Q_g - \theta\xi + \bar{\theta}\bar{\xi}. \tag{4.10}$$

They satisfy the same algebra in the extended PB (4.8) as $Q,\ \bar{Q},\ K,\ \bar{K}$ and Q_g satisfy in (2.21).

According to [9] we have an extended Hamiltonian flow defined by

$$\tilde{A}(t,\theta,\bar{\theta},\rho) = \exp\{Ad(F)\}A\exp\{-Ad(F)\}, \tag{4.11}$$

where

$$\begin{aligned} &F(t,\theta,\bar{\theta},\rho) \equiv -\{\Delta,\{\bar{\Delta},tH\}_e\}_e = \\ &= -t\{Q,\{\bar{Q},H\}\} + \theta\{\bar{Q},H\} - \bar{\theta}\{Q,H\} + \rho H, \end{aligned} \tag{4.12}$$

and where $Ad(F)$ is defined by

$$Ad(F) \equiv (F\overleftarrow{\partial_A})\omega^{AB}\partial_B, \tag{4.13}$$

where in turn ∂_A are derivatives with respect to Φ^A which represent all variables in the extended formulation ($\{\Phi^A,\Phi^A\} = \omega^{AB}$). Notice that the previous Hamiltonian flow is obtained from (4.11) if we impose $\theta = \bar{\theta} = \rho = 0$. In fact, any function A of ϕ^a satisfies the original Hamiltonian flow since $\tilde{A}(\phi,t)$ in (4.11) is independent of θ, $\bar{\theta}$ and ρ. The original equations (2.2) are not only obtained from (4.6), but also in terms of the Q-bracket in (4.7), as well as in terms of the extended bracket (4.8). One may also define a Δ-bracket defined by

$$\{A,B\}_\Delta \equiv \frac{1}{2}\Big(\{\{A,\bar{\Delta}\}_e,\{\Delta,B\}_e\}_e - \{\{A,\Delta\}_e,\{\bar{\Delta},B\}_e\}_e\Big), \tag{4.14}$$

which is an extended Q-bracket (cf (4.4)). The Δ-bracket is spanned by ϕ^a and t, σ. We have

$$\{\phi^a,\phi^b\}_\Delta = \omega^{ab}, \quad \{t,\sigma\}_\Delta = 1. \tag{4.15}$$

Obviously (cf (4.7))

$$\dot{\phi}^a = \{\phi^a,H\}_\Delta = \omega^{ab}\partial_b H. \tag{4.16}$$

5 Remarks on quantization

All formulas in the previous section may also be given in terms of operators and commutators after a canonical quantization defined by (3.4). There is then a Q-commutator defined by

$$[A,B]_Q \equiv \frac{1}{2}\Big([[A,\bar{Q}],[Q,B]] - [[A,Q],[\bar{Q},B]]\Big). \tag{5.1}$$

It follows that $[\phi^a,\phi^b]_Q$ satisfies all properties of a commutator. However, since it is *not* a commutator, ϕ^a as operators are still commuting. In fact, we have

$$[A(\phi),B(\phi)]_Q=-i\{A(\phi),B(\phi)\}_Q=-iA(\phi)\overleftarrow{\partial}_a\,\omega^{ab}\partial_b B(\phi). \tag{5.2}$$

Thus, even the operator formulation describes classical mechanics. It seems therefore as if the quantum BRST-antiBRST formulation of classical mechanics does not contain any new suggestion for quantization. (However, see [11] for an interesting proposal.) A more conventional constraint formulation of classical mechanics seems therefore to be more hopeful for the quantization of mechanics on general symplectic manifolds [12, 13].

A A further extended BRST-antiBRST algebra

The original Hamiltonian equation (2.2) cannot be derived from a Lagrangian written only in terms of ϕ^a. However, if the two-form Ω in (2.4) is exact then this is possible. An exact Ω is $\Omega=2d\phi^a\wedge df_a(\phi)$ which implies

$$\omega_{ab}=\partial_a f_b(\phi)+\partial_b f_a(\phi)(-1)^{(\varepsilon_a+1)(\varepsilon_b+1)}. \tag{A.1}$$

The Lagrangian $L=f_a(\phi)\dot{\phi}^a-H(\phi)$ yields then the equations $\omega_{ab}\dot{\phi}^b=\partial_a H$ with ω_{ab} given by (A.1) which are equal to (2.2) due to (2.3). In this case there is a natural extension of the algebra (2.21) (cf.[14]). This extension involves the following three new generators

$$\begin{aligned}&q\equiv\mathcal{C}^a f_a(\phi),\quad \bar{q}\equiv\mathcal{P}_a\omega^{ab}f_b(\phi)(-1)^{\varepsilon_b},\\ &D\equiv\{Q,\bar{q}\}=\lambda_a\omega^{ab}f_b(-1)^{\varepsilon_b}+\mathcal{P}_a\mathcal{C}^b\partial_b(\omega^{ac}f_c)(-1)^{\varepsilon_a+\varepsilon_b+\varepsilon_c}.\end{aligned} \tag{A.2}$$

The reality properties (2.12) requires $f_a^*=f_a(-1)^{\varepsilon_a}$ since $(\omega_{ab})^*=-\omega_{ba}$. The Lagrangian above is then real as well as the generators $\bar{q}$ and D while q is imaginary. The complete algebra is given by (2.21) together with

$$\begin{aligned}&\{q,q\}=\{q,\bar{q}\}=\{\bar{q},\bar{q}\}=\{K,q\}=\{\bar{K},\bar{q}\}=0,\\ &\{Q,q\}=-K,\quad \{\bar{Q},\bar{q}\}=\bar{K},\quad \{K,\bar{q}\}=q,\quad \{\bar{K},q\}=\bar{q},\\ &\{Q,\bar{q}\}=D,\quad \{\bar{Q},q\}=-D-Q_g,\quad \{Q,D\}=\{\bar{q},D\}=0,\\ &\{q,D\}=-q,\quad \{\bar{Q},D\}=\bar{Q},\quad \{K,D\}=-K,\quad \{\bar{K},D\}=\bar{K},\\ &\{q,Q_g\}=q,\quad \{\bar{q},Q_g\}=-\bar{q},\quad \{D,Q_g\}=0.\end{aligned} \tag{A.3}$$

The new generators (A.2) are, unfortunately, not conserved. This is therefore not a symmetry algebra.

B Properties of the Q-bracket (4.4)

Consider the Q-bracket (4.4), *i.e.*

$$\{A,B\}_Q\equiv\frac{1}{2}\Big(\{\{A,\bar{Q}\},\{Q,B\}\}-\{\{A,Q\},\{\bar{Q},B\}\}\Big). \tag{B.1}$$

As stated in section 3 this bracket satisfies all properties of a graded PB except for the Jacobi identities and Leibniz' rule. In fact, it satisfies the generalized Jacobi identities [9]

$$\begin{aligned}
&\{A,\{B,C\}_Q\}_Q(-1)^{\varepsilon_A\varepsilon_C} + cycle(A,B,C) = \\
&= \Big(2\{\{A,B\}_Q,\tilde{C}\} + \{\{\tilde{A},B\} + \{A,\tilde{B}\},C\}_Q\Big)(-1)^{\varepsilon_A\varepsilon_C} + cycle(A,B,C),
\end{aligned} \tag{B.2}$$

where the tilde functions are defined by

$$\tilde{f} \equiv \{Q,\{\bar{Q},f\}\}. \tag{B.3}$$

Instead of Leibniz' rule the Q-bracket (B.1) satisfies

$$\begin{aligned}
&\{A,BC\}_Q - \{A,B\}_Q C - B\{A,C\}_Q(-1)^{\varepsilon_A\varepsilon_B} = \\
&= -\{A,\{B,C\}_Q\} + \{\{A,B\},C\}_Q + \{B,\{A,C\}\}_Q(-1)^{\varepsilon_A\varepsilon_C}.
\end{aligned} \tag{B.4}$$

Notice that Leibniz' rule is satisfied on a subset of 'commuting' coordinates provided the Q-bracket closes on this subset.

The BRST-antiBRST quantum theory for pseudoclassical mechanics leads to a Q-bracket with the following elements

$$\begin{aligned}
&\{\phi^a,\phi^b\}_Q = \omega^{ab}, \quad \{\phi^a,\lambda_b\}_Q = -\frac{1}{2}\partial_b\omega^{ac}\lambda_c(-1)^{\varepsilon_b+\varepsilon_c+\varepsilon_a\varepsilon_b}, \\
&\{\phi^a,\mathcal{C}^b\}_Q = \frac{1}{2}\mathcal{C}^c\partial_c\omega^{ab}(-1)^{\varepsilon_a+\varepsilon_b+\varepsilon_c}, \quad \{\phi^a,\mathcal{P}_b\}_Q = \frac{3}{2}\mathcal{P}_c\partial_b\omega^{ca}(-1)^{\varepsilon_a+\varepsilon_b+\varepsilon_b(\varepsilon_a+\varepsilon_c)}, \\
&\{\lambda_a,\lambda_b\}_Q = \{\lambda_a,\mathcal{C}^b\}_Q = \{\mathcal{C}^a,\mathcal{C}^b\}_Q = \{\mathcal{P}_a,\mathcal{P}_b\}_Q = 0, \\
&\{\lambda_a,\mathcal{P}_b\}_Q = -\frac{1}{2}\mathcal{P}_c\partial_a\partial_b\omega^{cd}\lambda_d(-1)^{\varepsilon_b+\varepsilon_d+\varepsilon_c(\varepsilon_a+\varepsilon_b)} + \\
&\quad +\frac{1}{4}\mathcal{P}_d\mathcal{P}_e\mathcal{C}^c\partial_a\partial_b\partial_c\omega^{ed}(-1)^{\varepsilon_c+\varepsilon_e+(\varepsilon_a+\varepsilon_b)(\varepsilon_c+\varepsilon_d+\varepsilon_e)}, \\
&\{\mathcal{C}^a,\mathcal{P}_b\}_Q = -\frac{1}{2}\partial_b\omega^{ac}\lambda_c(-1)^{\varepsilon_b+\varepsilon_c+\varepsilon_a\varepsilon_b} + \frac{1}{2}\mathcal{P}_d\mathcal{C}^c\partial_c\partial_b\omega^{da}(-1)^{\varepsilon_b+\varepsilon_c+\varepsilon_d+\varepsilon_b(\varepsilon_a+\varepsilon_d)}.
\end{aligned} \tag{B.5}$$

The third line implies that the Q-bracket is trivial in the subsectors $\{\lambda_a\}$, $\{\mathcal{C}^a\}$ and $\{\mathcal{P}_a\}$. Only $\{\phi^a\}$ span the Q-bracket. Leibniz' rule is satisfied since the right-hand side of (B.4) is zero for functions of ϕ^a. In fact also the the Jacobi identities are satisfied. We have the following basic tilde operators:

$$\begin{aligned}
&\tilde{\phi}^a = -\lambda_b\omega^{ba} - \mathcal{P}_b\mathcal{C}^c\partial_c\omega^{ba}(-1)^{\varepsilon_b+\varepsilon_c}, \\
&\tilde{\lambda}_a = \lambda_b\partial_a\omega^{bc}\lambda_c(-1)^{\varepsilon_a+\varepsilon_c+\varepsilon_a\varepsilon_b} + \mathcal{P}_b\mathcal{C}^c\partial_c\partial_a\omega^{bd}\lambda_d(-1)^{\varepsilon_a+\varepsilon_b+\varepsilon_c+\varepsilon_d+\varepsilon_a\varepsilon_b}, \\
&\tilde{\mathcal{C}}^a = -2\mathcal{C}^c\partial_c\omega^{ab}\lambda_b(-1)^{\varepsilon_a+\varepsilon_b+\varepsilon_c} - \mathcal{P}_b\mathcal{C}^c\mathcal{C}^d\partial_d\partial_c\omega^{ba}(-1)^{\varepsilon_a+\varepsilon_d}, \\
&\tilde{\mathcal{P}}_a = \frac{1}{2}\mathcal{P}_d\mathcal{P}_b\mathcal{C}^c\partial_c\partial_a\omega^{bd}(-1)^{\varepsilon_a+\varepsilon_b+\varepsilon_a(\varepsilon_b+\varepsilon_d)} + \mathcal{P}_c\lambda_b\partial_a\omega^{bc}(-1)^{\varepsilon_a+\varepsilon_a(\varepsilon_b+\varepsilon_c)}.
\end{aligned} \tag{B.6}$$

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